ON HYPERSINGULAR INTEGRALS AND LEBESGUE SPACES OF DIFFERENTIABLE FUNCTIONS. II

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1. **Introduction.** The purpose of this paper is to extend our results obtained in [8]. We will be dealing with points $x=(x_1,\dots,x_n)$, $z=(z_1,\dots,z_n)$ of n-dimensional Euclidean space E^n and will use the standard notations $x+z=(x_1+z_1,\dots,x_n+z_n)$, $x\cdot z=x_1z_1+\cdots+x_nz_n$, $(x\cdot x)^{1/2}=|x|$, $\lambda x=(\lambda x_1,\dots,\lambda x_n)$ where λ is a real number, $dx=dx_1\cdots dx_n$, etc. If $x\neq 0$ then $x'=x/|x|$ is the projection of x onto the unit sphere Σ centered at the origin. For $1\leqq p<\infty$, L^p denotes the class of all functions f with $\|f\|_p=(\int_{E^n}|f(x)|^p\,dx)^{1/p}<+\infty$.

For $0<\alpha<2$ and $\varepsilon>0$, we consider the integral

$$\tilde{f}_\varepsilon(x) = \int_{|z|>\varepsilon} [f(x-z)-f(x)]\,\frac{\Omega(z')}{|z|^{n+\alpha}}\,dz$$

where $\Omega(z')$ is homogeneous of degree zero and satisfies

$$\int_\Sigma z_j'\Omega(z')\,dz' = 0$$

for each $j=1,\dots,n$ when $1\leqq\alpha<2$. If $f\in L^p$, $1\leqq p<\infty$, and $\hat{f}(x)=(1+|x|^2)^{-\alpha/2}\hat{f}_\alpha(x)$(1) for some $f_\alpha\in L^p$, we say $f\in L^p_\alpha$ and write $\|f\|_{p,\alpha}=\|f_\alpha\|_p$ (see for example [1]).

We showed in [8] that if $f\in L^p_\alpha$, $1<p<\infty$, $0<\alpha<2$, and Ω is infinitely differentiable on Σ then $\tilde{f}_\varepsilon$ converges in L^p to a function $\tilde{f}$ and $\|\tilde{f}\|_p\leqq c_{p,\alpha}\|f\|_{p,\alpha}$. The same result was also true when $p=1$ provided $\Omega\equiv 1$(2). The purpose of this paper is to obtain the same conclusions for $1<p<\infty$ under less restrictive hypotheses on Ω and to obtain a weak-type result for $p=1$ when Ω is not identically 1. It turns out that it is enough to assume in either case that Ω is merely integrable on Σ.

In the final analysis our method is to examine the behavior of $\tilde{f}_\varepsilon$ in the case $n=1$ and to apply the results obtained there to the case of any dimension using in part the "method of rotation" from singular integrals (see [3]). We note here that when $n=1$ the only essential choices of $\Omega(z)$ satisfying our conditions are $\Omega(z)\equiv 1$ or $\Omega(z)=\operatorname{sign} z$ when $0<\alpha<1$ and $\Omega(z)\equiv 1$ when $1\leqq\alpha<2$. For $n\geqq 1$ we will prove the following theorems.

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(1) $\hat{f}$ and $\hat{f}_\alpha$ denote the Fourier transforms of f and f_α in the sense of distributions.

(2) The case $\Omega\equiv 1$ was considered earlier by Stein [6].

THEOREM 1. *Let $f \in L^p_\alpha$, $1 \leqq p < \infty$, $0 < \alpha < 2$, and let $\Omega(z')$ be homogeneous of degree zero and integrable on Σ, and satisfy*

$$\int_\Sigma z'_j \Omega(z')\, dz' = 0 \qquad (j = 1, \ldots, n)$$

when $1 \leqq \alpha < 2$. Let

$$\tilde{f}_\varepsilon(x) = \int_{|z| > \varepsilon} [f(x-z) - f(x)] \frac{\Omega(z')}{|z|^{n+\alpha}}\, dz.$$

Then if $1 < p < \infty$, $\tilde{f}_\varepsilon(x)$ converges in L^p to a function $\tilde{f}(x)$ and

$$\|\tilde{f}_\varepsilon(x)\|_p \leqq c\|f\|_{p,\alpha},$$

where c is a constant independent of ε and f. In particular

$$\|\tilde{f}\|_p \leqq c\|f\|_{p,\alpha}.$$

If $p = 1$ and $s > 0$,

$$|\{x : |\tilde{f}_\varepsilon(x)| > s\}| \leqq (c/s)\|f\|_{1,\alpha},$$

with c independent of ε and f.

THEOREM 2. *Under the same hypotheses as in Theorem 1, $\tilde{f}_\varepsilon$ converges pointwise almost everywhere to a function $\tilde{f}$ if $1 \leqq p < \infty$. When $p = 1$*

$$|\{x : |\tilde{f}(x)| > s\}| \leqq (c/s)\|f\|_{1,\alpha}.$$

THEOREM 3. *Under the same hypotheses as in Theorem 1,*

$$\|f^*\|_p \leqq c\|f\|_{p,\alpha}, \qquad 1 < p < \infty,$$

and

$$|\{x : |f^*(x)| > s\}| \leqq (c/s)\|f\|_{1,\alpha}$$

where $f^*(x) = \sup_{\varepsilon > 0} |\tilde{f}_\varepsilon(x)|$.

We will first prove Theorem 1 for $n \geqq 1$. We use Theorem 1 for $n = 1$ to prove Theorem 3 for $n = 1$ and $1 < p < \infty$. We obtain Theorem 3 for $n > 1$ from the case $n = 1$ and a rotation argument(3). In general, we will consider only the range $1 \leqq \alpha < 2$ with comments on any changes needed when $0 < \alpha < 1$.

2. Theorem 1 for $p = 2$.

LEMMA 1. *If $\alpha > 0$ then*
(a) *$|x|^\alpha = (1 + |x|^2)^{\alpha/2}\, d\hat{\mu}(x)$ and*

(3) One further comment is in order here. Theorem 1 can be obtained as a corollary of Theorem 2 of *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 356–365, of Benedek, Calderón and Panzone. Since, however, the case $p = 1$ is not treated explicitly there, we have chosen instead the technique of [2]. Both methods, of course, amount to the same thing.

(b) $(1+|x|^2)^{\alpha/2} = |x|^\alpha \, d\hat{\sigma}(x) + d\hat{\tau}(x)$,
where $d\hat{\mu}$, $d\hat{\sigma}$ *and* $d\hat{\tau}$ *are the Fourier transforms of finite measures* $d\mu$, $d\sigma$ *and* $d\tau$.
See [6].

Suppose $f \in \mathcal{D}$ (the space of infinitely differentiable functions with compact support), $1 \leqq \alpha < 2$, $\Omega \in L^1(\Sigma)$ and

$$\int_\Sigma z_j' \Omega(z')\, dz' = 0 \qquad (j = 1, \ldots, n).$$

If

$$\tilde{f}_\varepsilon(x) = \int_{|z|>\varepsilon} [f(x-z) - f(x)] \frac{\Omega(z')}{|z|^{n+\alpha}}\, dz$$

then

$$\begin{aligned} \tilde{f}_\varepsilon\hat{\ }(x) &= \hat{f}(x) \int_{|z|>\varepsilon} \frac{\Omega(z')}{|z|^{n+\alpha}} [\exp[-i(x \cdot z)] - 1]\, dz \\ &= |x|^\alpha \hat{f}(x) B_\varepsilon(x) \qquad (x \neq 0), \end{aligned}$$

where

$$B_\varepsilon(x) = \int_\Sigma \Omega(z')\, dz' \int_{\varepsilon|x|}^\infty [\exp[-is(x' \cdot z')] - 1] \frac{ds}{s^{\alpha+1}}.$$

Here

$$\int_\Sigma |\Omega(z')|\, dz' \int_1^\infty |\exp[-is(x' \cdot z')] - 1| \frac{ds}{s^{\alpha+1}} \leqq 2 \int_\Sigma |\Omega(z')|\, dz' \int_1^\infty \frac{ds}{s^{\alpha+1}} < \infty,$$

while if $\varepsilon|x| < 1$ and $x \neq 0$,

$$\begin{aligned} &\left| \int_\Sigma \Omega(z')\, dz' \int_{\varepsilon|x|}^1 [\exp[-is(x' \cdot z')] - 1] \frac{ds}{s^{\alpha+1}} \right| \\ &\qquad = \left| \int_\Sigma \Omega(z')\, dz' \int_{\varepsilon|x|}^1 [\exp[-is(x' \cdot z')] - 1 + is(x'z')] \frac{ds}{s^{\alpha+1}} \right| \\ &\qquad \leqq c \int_\Sigma |\Omega(z')|\, dz' \int_0^1 s^2 \frac{ds}{s^{\alpha+1}} \end{aligned}$$

by the mean-value theorem. Since $\alpha < 2$, the last integral is finite. Hence $B(x) = \lim_{\varepsilon \to 0} B_\varepsilon(x)$ exists for $x \neq 0$ and $|B_\varepsilon(x)| \leqq B$ for $x \neq 0$. Moreover since $B_{\varepsilon\lambda}(x) = B_\varepsilon(\lambda x)$ for $\lambda > 0$, $B(x)$ is homogeneous of degree zero. Since

$$\tilde{f}_\varepsilon\hat{\ }(x) = (1+|x|)^{\alpha/2} (f * d\mu)\hat{\ }(x) B_\varepsilon(x),$$

it follows from Plancherel's formula that

$$\|\tilde{f}_\varepsilon\|_2 \leqq c\|f\|_{2,\alpha},$$

and that $\tilde{f}_\varepsilon$ converges in the L^2 norm as $\varepsilon \to 0$. That the same is true for any $f \in L^2_\alpha$ follows from the fact that $\mathcal{D}$ is dense in L^2_α.

3. **Theorem 1 for** $p=1$.

LEMMA 2. *If* $f\in L^p_\alpha$, $1\leqq p<\infty$, *then* $f=G_\alpha * f_\alpha$ *where* $f_\alpha\in L^p$, $\|f\|_{p,\alpha}=\|f_\alpha\|_p$, *and* G_α *has the following properties*:

(a) $G_\alpha\geqq 0$, $G_\alpha\in L^1(E^n)$,

(b) $G_\alpha(x)\leqq c_\alpha|x|^{\alpha-n}$ *if* $n>\alpha$,

(c) $|(\partial^\nu/\partial x^\nu)G_\alpha(x)|\leqq c_{\alpha,\nu}|x|^{\alpha-|\nu|-n}$ *if* $|\nu|>0$ *and* $n+1>\alpha$.

For a proof, see for example [4].

LEMMA 3. *Let* $h\geqq 0$, $h\in L^1(E^n)$. *Given* $s>0$, *there is a set* D (*depending on* s) *and a corresponding decomposition* $h=g+b$ *such that*

(a) D *is the union of nonoverlapping cubes* I_k *and* $|D|=\sum_k|I_k|\leqq s^{-1}\|h\|_1$,

(b) $g\in L^2\cap L^1$, $\|g\|_1\leqq\|h\|_1$, $\|g\|_2^2\leqq cs\|h\|_1$,

(c) $b=0$ *outside* D *and* $\int_{I_k}b(x)\,dx=0$ *for all* k.

For a proof, see [2, p. 91–94].

If $f\in L^1_\alpha$, write $f=G_\alpha * f_\alpha, f_\alpha\in L^1$. We may assume $f_\alpha\geqq 0$ without loss of generality. Given $s>0$, form the decomposition $f_\alpha=g_\alpha+b_\alpha$ of Lemma 3 so that $f=(G_\alpha * g_\alpha)+(G_\alpha * b_\alpha)=g+b$ and $\tilde f_\varepsilon=\tilde g_\varepsilon+\tilde b_\varepsilon$. Since $g\in L^2_\alpha$, we have

$$|\{x:|\tilde g_\varepsilon(x)|>s/2\}|\leqq (2/s)^2\|\tilde g_\varepsilon\|_2^2\leqq (2/s)^2\|g_\alpha\|_2^2\leqq (c/s)\|g_\alpha\|_1\leqq (c/s)\|f\|_{1,\alpha}.$$

Since $|D|\leqq s^{-1}\|f\|_\alpha=s^{-1}\|f\|_{1,\alpha}$, it remains only to prove that

$$\left|\left\{x\notin\bigcup I_k^*:|\tilde b_\varepsilon(x)|>s/2\right\}\right|\leqq (c/s)\|f\|_{1,\alpha}$$

where I_k^* is a cube concentric with I_k whose edge length is a suitably large fixed multiple of the edge length of I_k. In particular it is enough to show that

$$\int_{(\bigcup I_k^*)'}|\tilde b_\varepsilon(x)|\,dx\leqq c\|b_\alpha\|_1 \tag{3.1}$$

with c independent of ε. Writing

$$b(x-z)-b(x)=\int b_\alpha(y)[G_\alpha(x-z-y)-G_\alpha(x-y)]\,dy$$

and interchanging the order of integration (the iterated integrals converge absolutely for almost all x), we obtain

$$\tilde b_\varepsilon(x)=\sum_k\int_{I_k}b_\alpha(y)\,dy\int_{|z|>\varepsilon}[G_\alpha(x-y-z)-G_\alpha(x-y)]\frac{\Omega(z')}{|z|^{n+\alpha}}\,dz$$

for almost all x. Since $\int_{I_k}b_\alpha(y)\,dy=0$, this expression is unchanged if we replace $G_\alpha(x-y-z)-G_\alpha(x-y)$ in the integral above by $G_\alpha(x-y-z)-G_\alpha(x-y)-G_\alpha(x-y_k-z)+G_\alpha(x-y_k)$, where y_k is the center of I_k. Hence (3.1) will follow if for $|y-y_k|<d_k$

$$\int_{|x-y_k|>\lambda d_k}dx\left|\int_{|z|>\varepsilon}[G_\alpha(x-y-z)-G_\alpha(x-y)-G_\alpha(x-y_k-z)+G_\alpha(x-y_k)]\frac{\Omega(z')}{|z|^{n+\alpha}}\,dz\right|$$

is bounded independently of ε, d_k being the edge length of I_k. Changing variables, it is enough to show that for $|y|<d$

$$\int_{|x|>\lambda d} dx \left| \int_{|z|>\varepsilon} [G_\alpha(x-y-z)-G_\alpha(x-y)-G_\alpha(x-z)+G_\alpha(x)] \frac{\Omega(z')}{|z|^{n+\alpha}}\, dz \right|$$

is bounded by a constant independent of ε. This fact was essentially proved in [9] under the hypothesis that $\Omega \in L^1(\Sigma)$ and $n \geqq 2$. However, since the present form is slightly different and since (due to the Lemma 2(b)) there is a technical difficulty for $n=1$ and $1 \leqq \alpha < 2$, we will give a proof. Write the inner integral as

$$\int [G_\alpha(x-y-z)-G_\alpha(x-y)-G_\alpha(x-z)+G_\alpha(x)] \frac{\Omega(z')}{|z|^{n+\alpha}} \chi_\varepsilon(z)\, dz$$
$$= \int_{|z|<d} + \int_{|z|>d} = A_\varepsilon(x,y)+B_\varepsilon(x,y),$$

where $\chi_\varepsilon(z)$ is the characteristic function of $|z|>\varepsilon$. Since Ω is orthogonal to polynomials of degree 1, A_ε is unchanged if we add

$$\sum_{j=1}^{n} z_j G_\alpha^{(j)}(x-y) - \sum_{j=1}^{n} z_j G_\alpha^{(j)}(x)$$

inside the square brackets of its integrand, where $G_\alpha^{(j)} = \partial G_\alpha/\partial x_j$. Since $|y|<d$, $|x|>\lambda d$ (λ large), and $|z|<d$, we have by the mean-value theorem and Lemma 2(c) ($\alpha<2\leqq n+1$) that

$$|A_\varepsilon| \leqq \frac{c}{|x|^{n+2-\alpha}} \int_{|z|<d} |z|^2 \frac{|\Omega(z')|}{|z|^{n+\alpha}}\, dz = cd^{2-\alpha}|x|^{-n-2+\alpha},$$

whose integral over $|x|>\lambda d$ is bounded.

$$|B_\varepsilon| \leqq \int_{|z|>d} |G_\alpha(x-y-z)-G_\alpha(x-y)-G_\alpha(x-z)+G_\alpha(x)| \frac{|\Omega(z')|}{|z|^{n+\alpha}}\, dz$$
$$= \int_{d<|z|<|x|/2} + \int_{|z|>|x|/2} = B_1+B_2.$$

The integrand of B_1 is majorized by $|\Omega(z')|/|z|^{n+\alpha}$ times

$$\left| G_\alpha(x-y-z)-G_\alpha(x-z)+\sum y_j G_\alpha^{(j)}(x-z) \right| + \left| G_\alpha(x-y)-G_\alpha(x)+\sum y_j G_\alpha^{(j)}(x) \right|$$
$$+\sum |y_j|\, |G_\alpha^{(j)}(x-z)-G_\alpha^{(j)}(x)|.$$

Since $|z|<|x|/2$ in B_1 and $|y|<d\leqq |x|/\lambda$,

$$B_1 \leqq \frac{c}{|x|^{n+2-\alpha}} \int_{d<|z|<|x|/2} (d^2+d|z|) \frac{|\Omega(z')|}{|z|^{n+\alpha}}\, dz$$
$$\leqq \frac{cd^2}{|x|^{n+2-\alpha}} \int_{|z|>d} \frac{|\Omega(z')|}{|z|^{n+\alpha}}\, dz + \frac{cd}{|x|^{n+2-\alpha}} \int_{d<|z|<|x|/2} \frac{|\Omega(z')|}{|z|^{n+\alpha-1}}\, dz.$$

Here

$$\frac{d^2}{|x|^{n+2-\alpha}}\int_{|z|>d}\frac{|\Omega(z')|}{|z|^{n+\alpha}}\,dz = O(d^{2-\alpha}|x|^{-n-2+\alpha}),$$

whose integral over $|x|>\lambda d$ is bounded. Also

$$d\int_{|x|>\lambda d}\frac{dx}{|x|^{n+2-\alpha}}\int_{d<|z|<|x|/2}\frac{|\Omega(z')|}{|z|^{n+\alpha-1}}\,dz$$

$$\leqq d\int_{|z|<d}\frac{|\Omega(z')|}{|z|^{n+\alpha-1}}\,dz\int_{|x|>2|z|}\frac{dx}{|x|^{n+2-\alpha}} = cd\int_{|z|>d}\frac{|\Omega(z')|}{|z|^{n+1}}\,dz = O(1).$$

The part

$$|G_\alpha(x-y)-G_\alpha(x)|\int_{|z|>|x|/2}\frac{|\Omega(z')|}{|z|^{n+\alpha}}\,dz$$

of B_2 is majorized by $(d/|x|^{n+1-\alpha})O(|x|^{-\alpha})$, whose integral over $|x|>\lambda d$ is bounded. The remaining part of B_2 is

$$\int_{|z|>|x|/2}|G_\alpha(x-y-z)-G_\alpha(x-z)|\,\frac{|\Omega(z')|}{|z|^{n+\alpha}}\,dz$$

$$= \int_{|z|>|x|/2;\,|x-z|>2d} + \int_{|z|>|x|/2;\,|x-z|<2d} = B_2'+B_2''.$$

Since $|x-z|>2d$ in B_2' and $|y|<d$, it follows from the mean-value theorem that

$$B_2' \leqq cd\int_{|z|>|x|/2;\,|x-z|>2d}\frac{1}{|x-z|^{n+1-\alpha}}\,\frac{|\Omega(z')|}{|z|^{n+\alpha}}\,dz.$$

Here $|x-z|<|x|+|z|<3|z|$. Choosing $0<\delta<1$ (then $\delta<\alpha$),

$$B_2' \leqq cd\int_{|z|>|x|/2;\,|x-z|>2d}\frac{1}{|x-z|^{n+1-\delta}}\,\frac{|\Omega(z')|}{|z|^{n+\delta}}\,dz$$

whose integral over $|x|>\lambda d$ is majorized by a constant times

$$d\int_{|z|>\lambda d/2}\frac{|\Omega(z')|}{|z|^{n+\delta}}\,dz\int_{|x-z|>2d}\frac{dx}{|x-z|^{n+1-\delta}} = O(1).$$

Write $G_\alpha = G_{\alpha/2} * G_{\alpha/2}$ and

$$B_2'' \leqq \int_{|z|>|x|/2;\,|x-z|<2d}\frac{|\Omega(z')|}{|z|^{n+\alpha}}\,dz\int G_{\alpha/2}(u)|G_{\alpha/2}(x-y-z-u)-G_{\alpha/2}(x-z-u)|\,du$$

$$= \int G_{\alpha/2}(u)du\int_{|z|>|x|/2;\,|x-z|<2d}|G_{\alpha/2}(x-y-z-u)-G_{\alpha/2}(x-z-u)|\,\frac{|\Omega(z')|}{|z|^{n+\alpha}}\,dz.$$

If the exterior integral is extended over $|u|>kd$ for large fixed k, the resulting integral is majorized by

$$d\int_{|u|>kd}\frac{du}{|u|^{n-\alpha/2}}\int_{|z|>|x|/2;\,|x-z|<2d}\frac{1}{|u|^{n+1-\alpha/2}}\,\frac{|\Omega(z')|}{|z|^{n+\alpha}}\,dz.$$

Here we have used Lemma 2(b) and the fact that $\alpha/2<1\leqq n$. Integrating with respect to x over $|x|>\lambda d$, we obtain

$$d^{n+1}\int_{|u|>kd}\frac{du}{|u|^{2n+1-\alpha}}\int_{|z|>\lambda d/2}\frac{|\Omega(z')|}{|z|^{n+\alpha}}\,dz = O(1).$$

Finally,

$$\int_{|u|<kd}G_{\alpha/2}(u)\,du\int_{|z|>|x|/2;\,|x-z|<2d}G_{\alpha/2}(x-z-u)\frac{|\Omega(z')|}{|z|^{n+\alpha}}\,dz$$
$$\leqq c\int_{|u|<kd}\frac{du}{|u|^{n-\alpha/2}}\int_{|z|>|x|/2;\,|x-z|<2d}G_{\alpha/2}(x-z-u)\frac{|\Omega(z')|}{|z|^{n+\alpha}}\,dz.$$

The integral over $|x|>\lambda d$ of this is majorized by

$$c\int_{|u|<kd}\frac{du}{|u|^{n-\alpha/2}}\int_{|z|>\lambda d/2}\frac{|\Omega(z')|}{|z|^{n+\alpha}}\,dz\int_{|x-z-u|<(k+2)d}\frac{dx}{|x-z-u|^{n-\alpha/2}} = O(1).$$

The integral

$$\int_{|u|<kd}G_{\alpha/2}(u)\,du\int_{|z|>|x|/2;\,|x-z|<2d}G_{\alpha/2}(x-y-z-u)\frac{|\Omega(z')|}{|z|^{n+\alpha}}\,dz$$

can be treated in the same way. This completes the proof of Theorem 1 for $p=1$ when $1\leqq\alpha<2$. For $0<\alpha<1$, the changes necessary are indicated in [9].

4. **Theorem 1 for** $1<p<\infty$. If $T_\varepsilon f=\tilde{f}_\varepsilon$ and $J^\alpha f=f*G_\alpha$ then the linear operator $S_\varepsilon=T_\varepsilon J^\alpha$ satisfies

$$\|S_\varepsilon f\|_2 \leqq c\|f\|_2$$

and

$$|\{x : |(S_\varepsilon f)(x)| > s\}| \leqq (c/s)\|f\|_1$$

with c independent of ε and f. By the Marcinkiewicz interpolation theorem (see [10, Vol. II, p. 11]) it follows that

$$\|\tilde{f}_\varepsilon\|_p \leqq c\|f\|_{p,\alpha}$$

for $1<p\leqq 2$, with c independent of ε. Since $\tilde{f}_\varepsilon$ converges in L^p for $f\in\mathcal{D}$ an approximation argument shows it converges in L^p for any $f\in L^p_\alpha$, $1<p\leqq 2$. That the same facts hold for $1<p<\infty$ now follows easily by duality, the dual of L^p_α being $L^{p'}_{-\alpha}$, $1/p+1/p'=1$ (see [1, p. 39]).

5. **Theorem 3 for** $1<p<\infty$ **and** $n=1$. In this section we will prove Theorem 3 for $1<p<\infty$ when $n=1$. If $0<\alpha<1$, the only essential choices for Ω are then $\Omega\equiv 1$ and $\Omega(z)=\text{sign } z$, while if $1\leqq\alpha<2$, the only choice is $\Omega\equiv 1$. We will consider the case $1\leqq\alpha<2$. Let $f\in L^p_\alpha$, $1<p<\infty$, $1\leqq\alpha<2$, and let $\tilde{f}$ denote the limit in L^p of $\tilde{f}_\varepsilon$. Let $\phi(x)$ be an infinitely differentiable function with support in $|x|<1$ which

is even, nonnegative and satisfies $\int_{-\infty}^{\infty} \phi(x)\,dx=1$. It is easy to check that $\tilde{\phi}(x) =\lim_{\varepsilon\to 0} \tilde{\phi}_\varepsilon(x)$ is a bounded integrable function. For if x is well outside the support of ϕ then $\tilde{\phi}_\varepsilon(x)$ is independent of ε—say

$$\tilde{\phi}(x) = \int_{|z|>1} \phi(x-z)\,\frac{dz}{|z|^{\alpha+1}},$$

and so is bounded and integrable for such x. If x is in a bounded set, $|\tilde{\phi}_\varepsilon(x)| \leqq C$ by the mean-value theorem, and the same is true for $\tilde{\phi}$. In the same way, it is easy to see $\tilde{\phi}$ is continuous.

For any $H\in L^1$, let $H_\varepsilon(x)=\varepsilon^{-1}H(x/\varepsilon)$ for $\varepsilon>0$. The idea of the proof is to compare $\tilde{f}*\phi_\varepsilon$ to $\tilde{f}_\varepsilon$.

LEMMA 4. *If $f\in L^p_\alpha(-\infty,\infty)$, $1<p<\infty$, $1\leqq\alpha<2$, then*

$$(\tilde{f}*\phi_\varepsilon)(x) = \varepsilon^{-\alpha-1}\int_{-\infty}^{\infty} [f(x-z)-f(x)]\tilde{\phi}(z/\varepsilon)\,dz.$$

We claim first that $\tilde{f}*\phi_\varepsilon=\varepsilon^{-\alpha}f*(\tilde{\phi})_\varepsilon$. For if $f\in\mathscr{D}$, then by §2,

$$\tilde{f}^\wedge(x) = c|x|^\alpha \hat{f}(x) \quad\text{and}\quad \tilde{\phi}^\wedge(x) = c|x|^\alpha\hat{\phi}(x)$$

where $c=c_\alpha$, and the claim follows by taking Fourier transforms. For any $f\in L^p_\alpha$, we apply $\|\tilde{f}\|_p\leqq c\|f\|_{p,\alpha}$ and an approximation argument. Finally, since $\tilde{\phi}^\wedge$ is continuous, $\tilde{\phi}^\wedge(0)=0$. Hence $\int_{-\infty}^{\infty}(\tilde{\phi})_\varepsilon(x)\,dx=0$ and Lemma 4 follows.

For $z\neq 0$, let $K(z)=|z|^{-\alpha-1}$. Let $K_1(z)=K(z)$ for $|z|>1$, $K_1(z)=0$ for $|z|\leqq 1$. Then

$$\tilde{f}_\varepsilon(x) = \varepsilon^{-\alpha-1}\int_{-\infty}^{\infty} [f(x-z)-f(x)]K_1(z/\varepsilon)\,dz$$

and

$$(\tilde{f}*\phi_\varepsilon)(x)-\tilde{f}_\varepsilon(x) = \varepsilon^{-\alpha-1}\int_{-\infty}^{\infty} [f(x-z)-f(x)]\,\Delta(z/\varepsilon)\,dz$$

where $\Delta(z)=\tilde{\phi}(z)-K_1(z)$. We now observe that

(I) $\Delta(z)$ is bounded,

(II) $\Delta(z)$ is even, and

(III) $|\Delta(z)|\leqq c|z|^{-2-\alpha}$ for $|z|\geqq 2$.

I and II are clear since corresponding statements for $\tilde{\phi}$ and K_1 are true. For $|z|\geqq 2$,

$$\begin{aligned}\Delta(z) &= \int_{-\infty}^{\infty} \phi(z-y)K(y)\,dy-K(z) = \int_{-\infty}^{\infty}\phi(y)[K(z-y)-K(z)]\,dy\\ &= \int_{|y|<1}\phi(y)[K(z-y)-K(z)]\,dy.\end{aligned}$$

However, for $|z|>2$ and $|y|<1$,

$$|K(z-y)-K(z)| = \big|\,|z-y|^{-1-\alpha}-|z|^{-1-\alpha}\big| \leqq c|z|^{-2-\alpha}.$$

Returning to Theorem 3 for $n=1$, let $f\in L^p_\alpha$, $1<p<\infty$, $1\leqq\alpha<2$, and

$$|f^*(x)| = \sup_\varepsilon |\tilde{f}_\varepsilon(x)|.$$

Since $\|\sup_\varepsilon(\tilde{f}*\phi_\varepsilon)\|_p\leqq c\|\tilde{f}\|_p\leqq c\|f\|_{p,\alpha}$, $\|f^*\|_p\leqq c\|f\|_{p,\alpha}$ will follow from the corresponding inequality for the supremum over ε of

$$(5.1)\qquad \begin{aligned} |(\tilde{f}*\phi_\varepsilon)(x)-\tilde{f}_\varepsilon(x)| &= \varepsilon^{-\alpha-1}\left|\int_0^\infty [f(x+z)+f(x-z)-2f(x)]\,\Delta(z/\varepsilon)\,dz\right| \\ &\leqq \varepsilon^{-\alpha-1}\int_0^\infty |f(x+z)+f(x-z)-2f(x)|\,|\Delta(z/\varepsilon)|\,dz. \end{aligned}$$

By I and III we may assume Δ is decreasing and nonnegative in $(0,\infty)$ and

$$(5.2)\qquad \int_0^\infty t^\alpha\,\Delta(t)\,dt < +\infty.$$

Hence (5.1) is majorized by a constant times

$$\varepsilon^{-\alpha-1}\int_0^\infty \Delta\left(\frac{t}{\varepsilon}\right)\frac{d}{dt}[t^{\alpha+1}F_x(t)]\,dt,$$

where

$$(5.3)\qquad F_x(t) = t^{-\alpha-1}\int_0^t |f(x+z)+f(x-z)-2f(x)|\,dz.$$

Integrating by parts, (5.1) is majorized by a constant times

$$\varepsilon^{-\alpha-1}\,\Delta\left(\frac{t}{\varepsilon}\right)t^{\alpha+1}F_x(t)\Big|_0^\infty - \varepsilon^{-\alpha}\int_0^\infty t^{\alpha+1}F_x(t)\,d\Delta_\varepsilon(t).$$

The last integral is majorized by

$$-\sup_t F_x(t)\varepsilon^{-\alpha}\int_0^\infty t^{\alpha+1}\,d\Delta_\varepsilon(t) = -\sup_t F_x(t)\int_0^\infty t^{\alpha+1}\,d\Delta(t) = c\sup_t F_x(t),$$

by (5.2) and the fact that for large t

$$\Delta(t) \leqq ct^{-\alpha-2} = o(t^{-\alpha-1}) \qquad (t\to\infty).$$

Since then the integrated term above is zero, Theorem 3 for $n=1$ and $1<p<\infty$, $1\leqq\alpha<2$ will follow from

LEMMA 5. *Let $f\in L^p_\alpha(-\infty,\infty)$, $1<p<\infty$, $1\leqq\alpha<2$, and let $F_x(t)$ be defined by* (5.3). *Then*

$$\left\|\sup_t F_x(t)\right\|_p \leqq c\|f\|_{p,\alpha}$$

with c independent of f.

If $f=G_\alpha * f_\alpha$ then

$$F_x(t) \leqq t^{-\alpha-1}\int_{-\infty}^{\infty} |f_\alpha(x-y)|\,dy \int_0^t |G_\alpha(y+z)+G_\alpha(y-z)-2G_\alpha(y)|\,dy$$

$$= t^{-\alpha-1}\int_{|y|<2t} + t^{-\alpha-1}\int_{|y|>2t} = A_x(t)+B_x(t).$$

By the mean-value theorem and Lemma 2 of §3,

$$B_x(t) \leqq ct^{-\alpha-1}\int_{|y|>2t} |f_\alpha(x-y)|\,\frac{dy}{|y|^{3-\alpha}}\int_0^t z^2\,dz$$

$$= ct^{2-\alpha}\int_{|y|>2t} |f_\alpha(x-y)|\,\frac{dy}{|y|^{3-\alpha}}.$$

Let $M_x(s)=s^{-1}\int_{|y|<s}|f_\alpha(x-y)|\,dy$ so that by the Hardy-Littlewood maximal operator theorem (see $[10]_{\mathrm{II}}$, p. 306),

$$\left\|\sup_s M_x(s)\right\|_p \leqq c_p\|f_\alpha\|_p.$$

Then

$$B_x(t) \leqq ct^{2-\alpha}\int_{2t}^{\infty} s^{\alpha-3}\frac{d}{ds}[sM_x(s)]\,ds$$

$$= ct^{2-\alpha}\left[s^{\alpha-2}M_x(s)\Big|_{2t}^{\infty} + c\int_{2t}^{\infty} s^{\alpha-3}M_x(s)\,ds\right]$$

$$\leqq c\sup_s M_x(s).$$

Since $G_\alpha = G_{\alpha/2} * G_{\alpha/2}$,

$$\int_0^t |G_\alpha(y+z)+G_\alpha(y-z)-2G_\alpha(y)|\,dz$$

$$\leqq \int G_{\alpha/2}(u)\,du \int_0^t |G_{\alpha/2}(y+z-u)+G_{\alpha/2}(y-z-u)-2G_{\alpha/2}(y-u)|\,dz.$$

If we extend the exterior integration over $|u|>kt$ (k large) then since $|y|<2t$ and $|z|<t$, we obtain at most a constant times

$$\int_{|u|>kt}\frac{du}{|u|^{1-\alpha/2}}\int_0^t \frac{z^2}{|u|^{3-\alpha/2}}\,dz = O(t^\alpha).$$

Here we use Lemma 2 of §3 and the fact that $\alpha/2<1$. If instead we extend the exterior integration above over $|u|<kt$, we obtain at most a constant times

$$\int_{|u|<kt}\frac{du}{|u|^{1-\alpha/2}}\int_{|x|<ct}\frac{dx}{|x|^{1-\alpha/2}} = O(t^\alpha).$$

Hence

$$A_x(t) \leqq ct^{-1}\int_{|y|<2t}|f_\alpha(x-y)|\,dy.$$

In particular, $\sup_t F_x(t) \leqq c\sup_s M_x(s)$ and Lemma 5 follows.

6. **Theorem 3 for** $1<p<\infty$ **and** $n \geqq 2$. In this section we will use Theorem 3 for $1<p<\infty$ and $n=1$ to obtain Theorem 3 for $1<p<\infty$ and $n \geqq 2$. The method is by the "method of rotation" and is modeled after that given in [3] for singular integrals. There is, however, one difference for hypersingular integrals. For singular integrals, one considers first the case of odd kernels and then reduces the case of even kernels to that of odd kernels. For hypersingular integrals and $1 \leqq \alpha < 2$, say, we consider first the case of even kernels and then reduce the case of odd kernels to the study of singular integrals with odd kernels, rather than to hypersingular integrals with even kernels.

LEMMA 6. *Let $f \in L^p_\alpha(E^n)$, $1<p<\infty$. Let z' be any fixed unit vector in E^n and let ω denote a vector in the $n-1$ dimensional subspace of E^n orthogonal to z'. Writing $x=tz'+\omega$, $f(tz'+\omega)$ belongs to $L^p_\alpha(-\infty, \infty)$ as a function of t for almost all ω, and if $\|f(tz'+\omega)\|_{p,\alpha}$ is its L^p_α norm with respect to t, then*

$$\left(\int \|f(tz'+\omega)\|^p_{p,\alpha}\, d\omega\right)^{1/p} \leqq c_{p,\alpha}\|f\|_{p,\alpha}.$$

For a proof, see [7, p. 1044].

Now suppose that $f \in L^p_\alpha(E^n)$, $1<p<\infty$, $1 \leqq \alpha < 2$, $n \geqq 2$. Suppose $\Omega(z') \in L^1(\Sigma)$ and is even. Then

$$\begin{aligned} \tilde{f}_\varepsilon(x) &= \int_{|z|>\varepsilon} [f(x-z)-f(x)] \frac{\Omega(z')}{|z|^{n+\alpha}}\, dz \\ &= \tfrac{1}{2} \int_{|z|>\varepsilon} [f(x+z)+f(x-z)-2f(x)] \frac{\Omega(z')}{|z|^{n+\alpha}}\, dz. \end{aligned}$$

Changing to polar coordinates $z=\rho z'$, $\rho=|z|$, we obtain

$$\tilde{f}_\varepsilon(x) = \tfrac{1}{2} \int_\Sigma \Omega(z') \tilde{f}_\varepsilon(x, z')\, dz',$$

where

$$\tilde{f}_\varepsilon(x, z') = \int_{|\rho|>\varepsilon} [f(x-\rho z')-f(x)] \frac{d\rho}{|\rho|^{\alpha+1}}.$$

Hence

$$\sup_\varepsilon |\tilde{f}_\varepsilon(x)| \leqq \tfrac{1}{2} \int_\Sigma |\Omega(z')| \sup_\varepsilon |\tilde{f}_\varepsilon(x, z')|\, dz',$$

and Theorem 3 will follow from Minkowski's inequality if the L^p norm with respect to x of $\sup_\varepsilon |\tilde{f}_\varepsilon(x, z')|$ is majorized by a constant times $\|f\|_{p,\alpha}$ uniformly in z'. Writing $x=tz'+\omega$ as in Lemma 6,

$$\tilde{f}_\varepsilon(tz'+\omega, z') = \int_{|\rho|>\varepsilon} [f((t-\rho)z'+\omega)-f(tz'+\omega)] \frac{d\rho}{|\rho|^{\alpha+1}}.$$

By Theorem 3 for $n=1$, $\sup_\varepsilon |\tilde{f}_\varepsilon(tz'+\omega, z')|$ has L^p norm with respect to t majorized by a constant times the L^p_α norm of $f(tz'+\omega)$ with respect to t. Taking L^p norms with respect to ω and applying Lemma 6, the result follows.

Writing Ω as the sum of its even and odd parts, we will obtain Theorem 3 for $1<p<\infty$ if we now prove it in the case that Ω is odd(4). Hence let $1\leqq\alpha<2$, $n\geqq 2$, and $K(z)=\Omega(z')|z|^{-n-\alpha}$ for $z\neq 0$, where Ω is odd, orthogonal to polynomials of degree 1 on Σ and integrable on Σ. Let $\phi(t)$, $t\geqq 0$, be infinitely differentiable, $\phi(t)=0$ in $(0, \frac{1}{4})$ and $\phi(t)=1$ in $(\frac{3}{4}, \infty)$, and define

$$K_2(x) = \int_{E^n} \frac{1}{|x-z|^{n-\alpha}} K(z)\phi(|z|)\, dz. \tag{6.1}$$

$K_2(x)$ converges absolutely for almost all x and is locally integrable. In fact, K_2 is the Riesz fractional integral of the function $K(z)\phi(|z|)\in L^1(E^n)$.

Now consider

$$\begin{aligned} K_1(x) &= \int_{E^n} \frac{1}{|x-z|^{n-\alpha}} K(z)\, dz \\ &= \lim_{\delta\to 0} \int_{|z|>\delta} \frac{1}{|x-z|^{n-\alpha}} K(z)\, dz. \end{aligned} \tag{6.2}$$

We claim this integral converges for almost all x and is integrable over $a\leqq|x| \leqq b$ for every $0<a<b<\infty$. For

$$\begin{aligned} &\int_{\delta<|z|<(a/2)} \frac{1}{|x-z|^{n-\alpha}} K(z)\, dz \\ &\qquad = \int_{\delta<|z|<(a/2)} \left[\frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|x|^{n-\alpha}} + \Sigma z_j \frac{\partial}{\partial x_j} \frac{1}{|x|^{n-\alpha}}\right] K(z)\, dz \end{aligned}$$

since K is odd and orthogonal to polynomials of degree 1. However, $|z|<a/2$ implies $|x|>2|z|$ for $a\leqq|x|\leqq b$. By the mean-value theorem, the integral on the right is majorized by a constant times

$$|x|^{-n-2+\alpha} \int_{|z|<a/2} |z|^2 \frac{|\Omega(z')|}{|z|^{n+\alpha}}\, dz$$

which is bounded in $|x|\geqq a$.

$$\int_{|z|>a/2} \frac{1}{|x-z|^{n-\alpha}} |K(z)|\, dz = \int_{a/2<|z|<2b} + \int_{|z|>2b}.$$

The second integral on the right is bounded since $|x-z|>b$. In the first integral $|x|<b$ implies $|x-z|<3b$. Integrating over $a\leqq|x|\leqq b$ and changing the order of integration we obtain at most

$$\int_{|z|>a/2} |K(z)|\, dz \int_{|x-z|<3b} \frac{dx}{|x-z|^{n-\alpha}} < +\infty.$$

LEMMA 7. *Let Ω be an odd function satisfying the hypothesis of Theorem* 1. *Let*

(4) If $0<\alpha<1$, the method just given for even Ω also works for odd Ω, using the one-dimensional kernel sign z.

K_1 and K_2 be defined by (6.1) *and* (6.2). *Then K_1 and K_2 are odd, K_1 is homogeneous of degree $-n$,*

$$|K_2(x)| \leqq G(x') \quad \text{in } |x| \leqq 1, \qquad G \in L^1(\Sigma),$$

and

$$|K_1(x)-K_2(x)| \leqq c|x|^{-n-2+\alpha} \quad \text{in } |x| \geqq 1.$$

It is clear that K_1 and K_2 are odd and that K_1 is positively homogeneous of degree $-n$. If $|x| \geqq 1$,

$$\begin{aligned}
|K_1(x)-K_2(x)| &= \left| \int \frac{1}{|x-z|^{n-\alpha}} K(z)\phi(|z|)\,dz - \int \frac{1}{|x-z|^{n-\alpha}} K(z)\,dz \right| \\
&\leqq \int_{|z|<3/4} \left| \frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|x|^{n-\alpha}} + \Sigma z_j \frac{\partial}{\partial x_j}\frac{1}{|x|^{n-\alpha}} \right| |K(z)|\,|\phi(|z|)-1|\,dz \\
&\leqq c|x|^{-n-2+\alpha} \int_{|z|<3/4} |z|^2 |K(z)|\,dz = O(|x|^{-n-2+\alpha})
\end{aligned}$$

in $|x| \geqq 1$.

Now let $|x| \leqq 1$. If $\frac{1}{8} \leqq |x| \leqq 1$,

$$\begin{aligned}
|K_2(x)| &\leqq |K_1(x)| + |K_2(x)-K_1(x)| \\
&\leqq |K_1(x)| + \left| \int \frac{1}{|x-z|^{n-\alpha}} K(z)[\phi(|z|)-1]\,dz \right| \\
&\leqq |K_1(x)| + \int \left| \frac{1}{|x-z|^{n-\alpha}} - \chi(|z|)\left(\frac{1}{|x|^{n-\alpha}} - \Sigma z_j \frac{\partial}{\partial x_j} \frac{1}{|x|^{n-\alpha}} \right) \right| |K(z)|\,dz,
\end{aligned}$$

where $\chi(t)$ is the characteristic function of $(0, 1)$. Choose $\delta > 0$ so that $\alpha+\delta<2$. Then for $\frac{1}{8} \leqq |x| \leqq 1$,

$$\left| \frac{1}{|x-z|^{n-\alpha}} - \chi(|z|)\left(\frac{1}{|x|^{n-\alpha}} - \Sigma z_j \frac{\partial}{\partial x_j} \frac{1}{|x|^{n-\alpha}} \right) \right| \leqq c|z|^{\alpha+\delta}|x-z|^{-n+\alpha}.$$

This follows by considering for example the cases $|z|>1$, $|z|<1/16$ (where $|x| > 2|z|$), and $1/16<|z|<1$. Hence for $\frac{1}{8} \leqq |x| \leqq 1$,

$$|K_2(x)| \leqq A\left[|x|^n |K_1(x)| + |x|^{n-\alpha-\delta} \int \frac{|z|^{\alpha+\delta}}{|x-z|^{n-\alpha}} |K(z)|\,dz \right]$$

since $n-\alpha-\delta>0$ ($n \geqq 2$). The right side here is homogeneous of degree zero. If $|x| \leqq \frac{1}{8}$, then since $\phi(|z|)=0$ for $|z|<\frac{1}{4}$, K_2 is bounded in $|x| \leqq \frac{1}{8}$. Hence $|K_2|$ is majorized in $|x| \leqq 1$ by a function homogeneous of degree zero. It remains to show this function is integrable on $|x|=1$. Since K_1 is locally integrable away from the origin, consider

$$|x|^{n-\alpha-\delta} \int \frac{|z|^{\alpha+\delta}}{|x-z|^{n-\alpha}} |K(z)|\,dz,$$

and split the integral into the sum of three, extended over $|z|<1/16$, $1/16\leqq|z|\leqq 2$, and $|z|>2$. If $\frac{1}{8}\leqq|x|\leqq 1$, the first is majorized by a constant times

$$\int_{|z|<1/16}|z|^{\alpha+\delta}|K(z)|\,dz < +\infty,$$

and so is bounded. By Young's theorem, the second is integrable over $\frac{1}{8}\leqq|x|\leqq 1$ since K is integrable in $1/16\leqq|z|\leqq 2$. The third is less than a constant times

$$\int_{|z|<2}\frac{|\Omega(z)|}{|z|^{n+(n-\alpha-\delta)}}\,dz$$

in $\frac{1}{8}\leqq|x|\leqq 1$, and so is bounded there since $n-\alpha-\delta>0$. By homogeneity, the lemma follows.

If g is a bounded function with compact support then by the well-known Sobolev lemma (see [5]),

$$f(x) = \int\frac{g(z)}{|x-z|^{n-\alpha}}\,dz = (R_\alpha g)(x)$$

belongs to L^p for $1<p<(n/\alpha)$. For such f,

$$\tilde{f}_\varepsilon(x) = \int_{|x-z|>\varepsilon}K(x-z)f(z)\,dz$$

since K is odd. Write

$$\tilde{f}_\varepsilon(x) = \int K(x-z)\phi\left(\frac{|x-z|}{\varepsilon}\right)f(z)\,dz-\int_{|x-z|<\varepsilon}K(x-z)\phi\left(\frac{|x-z|}{\varepsilon}\right)f(z)\,dz.$$

Here

$$\begin{aligned}\int K(x-z)\phi\left(\frac{|x-z|}{\varepsilon}\right)f(z)\,dz &= \int K(x-z)\phi\left(\frac{|x-z|}{\varepsilon}\right)\left[\int\frac{g(y)}{|z-y|^{n-\alpha}}\,dy\right]dz\\ &= \int g(y)\left[\int\frac{1}{|z-y|^{n-\alpha}}K(x-z)\phi\left(\frac{|x-z|}{\varepsilon}\right)dz\right]dy.\end{aligned}$$

Interchanging the order of integration here is justified since the inner integral with K replaced by $|K|$ was shown earlier to be locally integrable. Since g is bounded and has compact support, the last integral converges absolutely. Hence

$$\begin{aligned}\tilde{f}_\varepsilon(x) &= \varepsilon^{-n}\int g(z)K_2\left(\frac{x-z}{\varepsilon}\right)dz-\int_{|x-z|<\varepsilon}K(x-z)\phi\left(\frac{|x-z|}{\varepsilon}\right)f(z)\,dz\\ &= \varepsilon^{-n}\int_{|x-z|<\varepsilon}g(z)K_2\left(\frac{x-z}{\varepsilon}\right)dz+\varepsilon^{-n}\int_{|x-z|>\varepsilon}g(z)K_1\left(\frac{x-z}{\varepsilon}\right)dz\\ &\quad+\varepsilon^{-n}\int_{|x-z|>\varepsilon}g(z)\left[K_2\left(\frac{x-z}{\varepsilon}\right)-K_1\left(\frac{x-z}{\varepsilon}\right)\right]dz\\ &\quad-\int_{\varepsilon/4<|x-z|<\varepsilon}K(x-z)\phi\left(\frac{x-z}{\varepsilon}\right)f(z)\,dz.\end{aligned}\tag{6.3}$$

If $g \in L^1 \cap C_0$, where C_0 is the class of continuous functions tending to zero at infinity, take continuous functions with compact support $g_n \to g$ in $L^1 \cap C_0$. Then $g_n \to g$ in all L^p and $f_n = R_\alpha g_n$ converge in some L^p to f. Since $g \in L^1 \cap C_0$, $R_\alpha g$ is defined, and since $g_n \to g$ in L^1, $R_\alpha g_n \to R_\alpha g$ in L^1_{loc}. Hence $f = R_\alpha g$ almost everywhere. Using the properties of K_1 and K_2 established in Lemma 7, it follows that (6.3) is valid for $f = R_\alpha g$, $g \in L^1 \cap C_0$. Now let $f = G_\alpha * f_\alpha$ where $f_\alpha \in L^1 \cap C_0$. By Lemma 1, $f = R_\alpha g$ where $g = \bar{f}_\alpha = f_\alpha * d\mu \in L^1 \cap C_0$, and (6.3) is true for such f if we replace g by $\bar{f}_\alpha$. For any $f \in L^p_\alpha$, $f = G_\alpha * f_\alpha$. Take $f_\alpha^{(n)} \in L^1 \cap C_0$ converging to f_α in L^p. Then $f^{(n)} = G_\alpha * f_\alpha^{(n)}$ converges to f in L^p and $\bar{f}_\alpha^{(n)}$ converges to $\bar{f}_\alpha$ in L^p. We obtain

$$\begin{aligned}\bar{f}_\varepsilon(x) &= \varepsilon^{-n} \int_{|x-z|<\varepsilon} \bar{f}_\alpha(z) K_2\left(\frac{x-z}{\varepsilon}\right) dz + \varepsilon^{-n} \int_{|x-z|>\varepsilon} \bar{f}_\alpha(z) K_1\left(\frac{x-z}{\varepsilon}\right) dz \\ &\quad + \varepsilon^{-n} \int_{|x-z|>\varepsilon} \bar{f}_\alpha(z) \left[K_2\left(\frac{x-z}{\varepsilon}\right) - K_1\left(\frac{x-z}{\varepsilon}\right)\right] dz \qquad (6.4) \\ &\quad - \int_{\varepsilon/4 \leqq |x-z| < \varepsilon} K(x-z)\phi\left(\frac{|x-z|}{\varepsilon}\right) f(z)\, dz \\ &= I^1_\varepsilon(x) + I^2_\varepsilon(x) + I^3_\varepsilon(x) - I^4_\varepsilon(x),\end{aligned}$$

for $f \in L^p_\alpha$, $f = G_\alpha * f_\alpha$, $\bar{f}_\alpha = f_\alpha * d\mu$.

Now $I^2_\varepsilon = \int_{|x-z|>\varepsilon} \bar{f}_\alpha(z) K_1(x-z)\, dz$ since K_1 is homogeneous of degree $-n$. Since K_1 is odd, it follows from the theory of singular integrals (see [3, p. 289]) that for $1 < p < \infty$

$$\left\| \sup_\varepsilon |I^2_\varepsilon(x)| \right\|_p \leqq c_p \|\bar{f}_\alpha\|_p \leqq c_p \|f\|_{p,\alpha}.$$

To estimate the remaining integrals, we recall the following lemma from [3, p. 291].

LEMMA 8. *Let $N(x)$ be nonnegative, homogeneous of degree zero, and integrable on Σ, and let $\psi(t)$, $t \geqq 0$, be decreasing, $\psi(|x|) \in L^1(E^n)$. If*

$$f^*(x) = \sup_{\varepsilon>0} \varepsilon^{-n} \int_{E^n} |f(z)| N(x-z) \psi\left(\frac{|x-z|}{\varepsilon}\right) dz$$

then

$$\|f^*\|_p \leqq c\|f\|_p, \qquad 1 < p < \infty,$$

with c independent of f.

Since $|x-z| < \varepsilon$ in I^1_ε, we have by Lemma 7 that

$$|I^1_\varepsilon(x)| \leqq \varepsilon^{-n} \int_{|x-z|<\varepsilon} |\bar{f}_\alpha(z)| G\left(\frac{x-z}{|x-z|}\right) dz.$$

Hence

$$\left\| \sup_\varepsilon |I^1_\varepsilon(x)| \right\|_p \leqq c\|\bar{f}_\alpha\|_p \leqq c\|f\|_{p,\alpha}.$$

Since $|x-z|>\varepsilon$ in I_ε^3,

$$|I_\varepsilon^3(x)| \leqq c\varepsilon^{-n}\int_{|x-z|>\varepsilon} |\bar{f}_\alpha(z)|\,|(x-z)\varepsilon^{-1}|^{-n-2+\alpha}\,dz.$$

Since $n+2-\alpha>n$,

$$\Big\|\sup_\varepsilon|I_\varepsilon^3(x)|\Big\|_p \leqq c\|f\|_{p,\alpha}$$

as before.

Finally,

$$\begin{aligned} I_\varepsilon^4(x) &= \int_{\varepsilon/4<|z|<\varepsilon} K(z)\phi\Big(\frac{|z|}{\varepsilon}\Big)f(x-z)\,dz \\ &= \int f_\alpha(x-y)\,dy\int_{\varepsilon/4<|z|<\varepsilon} K(z)\phi\Big(\frac{|z|}{\varepsilon}\Big)G_\alpha(y-z)\,dz \\ &= \int_{|y|<2\varepsilon}+\int_{|y|>2\varepsilon} = A_\varepsilon+B_\varepsilon. \end{aligned}$$

B_ε is unchanged if we replace $G_\alpha(y-z)$ by $G_\alpha(y-z)-G_\alpha(y)+\Sigma z_j G_\alpha^{(j)}(y)$. Since $|y|>2\varepsilon>2|z|$,

$$\begin{aligned} |B_\varepsilon| &\leqq c\int_{|y|>2\varepsilon}|f_\alpha(x-y)|\,\frac{dy}{|y|^{n+2-\alpha}}\int_{|z|<\varepsilon}|z|^2\,\frac{|\Omega(z')|}{|z|^{n+\alpha}}\,dz \\ &= c\varepsilon^{2-\alpha}\int_{|y|>2\varepsilon}|f_\alpha(x-y)|\,\frac{dy}{|y|^{n+2-\alpha}}. \end{aligned}$$

Introducing the Hardy-Littlewood maximal operator for f_α and arguing as for the term $B_x(t)$ of Lemma 5, we obtain

$$\Big\|\sup_\varepsilon|B_\varepsilon|\Big\|_p \leqq c\|f_\alpha\|_p=c\|f\|_{p,\alpha}.$$

Now

$$|A_\varepsilon| \leqq c\int_{|y|<2\varepsilon}|f_\alpha(x-y)|\,dy\int_{\varepsilon/4<|y-z|<\varepsilon}|K(y-z)|G_\alpha(z)\,dz.$$

In the inner integral, $|z|\leqq|y|+|y-z|<3\varepsilon$. Interchanging the order of integration,

$$\begin{aligned} |A_\varepsilon| &\leqq c\varepsilon^{-n-\alpha}\int_{|z|<3\varepsilon}G_\alpha(z)\,dz\int_{|y|<\varepsilon}|f_\alpha(x-z-y)|\,|\Omega(y')|\,dy \\ &\leqq c\varepsilon^{-\alpha}\int_{|z|<3\varepsilon}G_\alpha(z)M(x-z)\,dz \end{aligned}$$

where

$$M(u) = \sup_\varepsilon \varepsilon^{-n}\int_{|y|<\varepsilon}|f_\alpha(u-y)|\,|\Omega(y')|\,dy.$$

Since $n\geqq 2>\alpha$,

$$\begin{aligned} |A_\varepsilon| &\leqq c\varepsilon^{-\alpha}\int_{|z|<3\varepsilon}|z|^{-n+\alpha}M(x-z)\,dz \\ &= c\varepsilon^{-n}\int_{|z|<3\varepsilon}(|z|\varepsilon^{-1})^{-n+\alpha}M(x-z)\,dz. \end{aligned}$$

Therefore,

$$\Big\|\sup_{\varepsilon} A_\varepsilon\Big\|_p \leqq c\|M\|_p \leqq c\|f_\alpha\|_p = c\|f\|_{p,\alpha}$$

by applying Lemma 5 twice. This completes the proof of Theorem 3 for $1<p<\infty$.

7. **Theorem 3 for** $p=1$, $n\geqq 1$. By a standard method, the conclusions of Theorem 3 imply those of Theorem 2. It remains therefore only to prove Theorem 3 when $p=1$.

Let $f\in L^1_\alpha$, $1\leqq\alpha<2$, and let Ω satisfy the hypotheses of Theorem 3. If

$$f^*(x) = \sup_{\varepsilon} |\tilde{f}_\varepsilon(x)|,$$

we must show that

$$|\{x : f^*(x) > s\}| \leqq (c/s)\|f\|_{1,\alpha}. \tag{7.1}$$

We form the decomposition $f=g+b$ as in §3. Then $f^*\leqq g^*+b^*$ and it is enough to prove (7.1) for g and b. Since $g\in L^2_\alpha$ and $\|g\|^2_{2,\alpha}\leqq s\|f\|_{1,\alpha}$, it is enough to prove (7.1) for b^*, or what amounts to the same thing to show

$$\int_{(\cup I_k^*)'} b^*(x)\,dx \leqq c\|b_\alpha\|_1.$$

In the notation of §3, it is then enough to show that

$$\int_{|x|>\lambda d}\Big(\sup_{\varepsilon}|A_\varepsilon(x,y)|+\sup_{\varepsilon}|B_\varepsilon(x,y)|\Big)\,dx$$

is bounded for $|y|<d$. We refer the reader to the proof given in §3.

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